

Indian Statistical Institute, Bangalore Centre
B.Math. (III Year) : 2009-2010
Semester I : Midsemestral Examination
Introduction to Differential Equations

September 2009 Time: $2\frac{1}{2}$ hrs. Max. Marks : 80
Note: This paper carries 83 marks. Any score above 80 will be taken as 80.

1. [10 marks] Assume that a particular type of bacteria is in an environment where there is no obstacle to growth. Suppose that the population grows at a rate proportional to the current population. Find the time required for the population to double in size.

2. [9+3 marks] (i) Find the general solution to

$$x'(t) + ax(t) = b \exp(-\lambda t), \quad t > 0,$$

where λ, a, b are real numbers.

(ii) If $\lambda > 0, a > 0$ in (i) above, what can you say about $\lim_{t \rightarrow \infty} x(t)$?

3. [12+3 marks] (i) Find the general solution to

$$x''(t) + 10x'(t) + 25x(t) = \exp(-5t),$$

for $t > 0$.

(ii) What can you say about $\lim_{t \rightarrow \infty} x(t)$ in (i) above?

4. [15 marks] Solve the problem :

$$\begin{aligned} x''(t) - 2x'(t) &= 12t - 10, \quad t > 0, \\ x(0) &= 0, \quad x'(0) = 0. \end{aligned}$$

5. [9+12+10 marks] Consider the ODE

$$x''(t) + q(t)x(t) = 0, \quad t > 0,$$

where $q(\cdot)$ is a continuous function on $[0, \infty)$.

(i) Let $\varphi_1(\cdot), \varphi_2(\cdot)$ be two linearly independent solutions to the ODE. Show that $W(\varphi_1, \varphi_2; \cdot)$ is a nonzero constant.

(ii) Suppose every solution to the ODE is thrice differentiable on $(0, \infty)$. Show that $q(\cdot)$ is differentiable on $(0, \infty)$.

(iii) Suppose $q(\cdot) < 0$ on $[0, \infty)$. Show that any nontrivial solution has at most one zero in $[0, \infty)$.